An investigation and analysis of constrained Hamiltonian approaches to general relativity: Emmy Noether revisited - again

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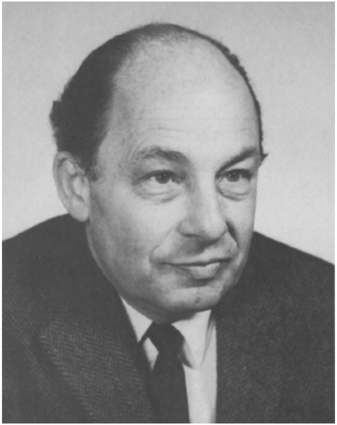

In Tribute to Peter G. Bergmann
Born in Berlin on March 24, 1915

Overview

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1. INTRODUCTION

Introduction

Focus of this talk:

- History of use of Noether's second theorem in constructing canonical generators of diffeomorphism transformations

Questions to be addressed:

- What is the relation between Emmy Noether's second theorem of 1918 and the general diffeomorphism symmetry of GR?
- What was the historical process whereby the implications of the theorem were recognized and exploited?
- What were the motivations and hopes of the historical actors in this drama?
- How and in what manner did diffeomorphism covariance cease to remain a foundational principle in the phase space description of general relativity - according to conventional wisdom?

2. EINSTEIN, NOETHER, KLEIN AND PAULI

The fundamental identity

Suppose the Lagrangian $\mathcal{L}$ transforms as a scalar density under the change of coordinates $x'^{\mu} = x^{\mu} + \delta x^{\mu}$ and depends on no higher than first derivatives $y_{A,\mu}$.

Let $\bar{\delta} y_A(x) := y'_A(x) - y_A(x) = \delta y_A(x) - y_{A,\mu}(x)\delta x^{\mu}$ represent the corresponding active change in the field variables. Then

$$\bar{\delta}\mathcal{L} \equiv -\left(\mathcal{L}\delta x^{\mu}\right)_{,\mu} \equiv \frac{\delta \mathcal{L}}{\delta y_A}\bar{\delta} y_A + \left(\frac{\partial \mathcal{L}}{\partial y_{A,\mu}}\bar{\delta} y_A\right)_{,\mu} \qquad \text{(I)}$$

Noether's 2nd theorem

Let

$$\bar{\delta} y_A = c_{A\mu}{}^{\nu}(y)\delta x^{\mu}_{,\nu} - y_{A,\mu}\delta x^{\mu}$$

Note: Noether considered wider symmetries including what we now call internal gauge. She put stress not only on the Lie algebra but also on the finite symmetry group elements. [Noether, 1918]

Employ these symmetry variations in the action:

$$\bar{\delta} S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta y_A} \bar{\delta} y_A + \left(\frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \bar{\delta} y_A \right)_{,\mu} \right] \equiv - \int d^4x \left(\mathcal{L} \delta x^{\mu} \right)_{,\mu}$$

Noether's 2nd theorem

Write as

$$0 \equiv \int d^4x \left[\frac{\delta \mathcal{L}}{\delta y_A} \bar{\delta} y_A + C^{\mu}_{,\mu} \right]$$

Substitute for $\bar{\delta} y_A$ and integrate by parts, letting $\delta x^{\mu} \to 0$ on the boundary, to get the generalization of the contracted Bianchi identities

$$\left(c_{A\mu}{}^{\nu} \frac{\delta \mathcal{L}}{\delta y_A} \right)_{,\nu} + \frac{\delta \mathcal{L}}{\delta y_A} y_{A,\mu} \equiv 0 \qquad \text{(II)}$$

(Derived by Pauli in his 1921 Relativity encyclopedia article) [Pauli, 1921]. Cites Klein, not Noether.)

The Noether current

Rewrite fundamental identity as

$$0 \equiv \frac{\delta \mathcal{L}}{\delta y_A} \bar{\delta} y_A + C^{\mu}_{,\mu} \tag{IIIA}$$

where

$$C^{\mu} := \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \bar{\delta} y_A + \mathcal{L} \delta x^{\mu} = \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \left(\delta y_A - y_{A,\nu} \delta x^{\nu} \right) + \mathcal{L} \delta x^{\mu} \tag{IIIB}$$

$$= \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \delta y_A - t^{\mu}{}_{\nu} \delta x^{\nu}. \tag{IIIC}$$

The stress-energy pseudo tensor is defined as

$$t^{\mu}{}_{\nu} := \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} y_{A,\nu} - \mathcal{L} \delta^{\mu}_{\nu} \tag{IIID}$$

Klein identities

Expand the fundamental identity in factors of derivatives of δx^μ:

$$0 \equiv \mathcal{U}_\mu^{\alpha\beta} \delta x^\mu_{,\alpha\beta} + \mathcal{U}_\mu^\alpha \delta x^\mu_{,\alpha} + \mathcal{U}_\mu \delta x^\mu$$

In particular, the coefficient of δx^ν gives

$$\frac{\delta \mathcal{L}}{\delta y_A} y_{A,\nu} + t^\mu{}_{\nu,\mu} \equiv 0 \qquad \text{(IV)}$$

Primary constraints

Coefficent of $\delta x^{\mu}_{,00}$ gives

$$0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}_A} c_{A\mu}{}^{0} = p^A c_{A\mu}{}^{0} =: \phi_\mu(y, p) \qquad \text{(V)}$$

3. LÉON ROSENFELD (1930)

Pauli and Rosenfeld

Pauli and Heisenberg were concerned about their treatment of the vanishing momentum in their pioneering work on quantum electrodynamics. See Salisbury 2009 for details of Rosenfeld's 1929 Zurich position with Pauli [Salisbury, 2009]

Rosenfeld's unified Lagrangian

Rosenfeld's Lagrangian yields Einstein's gravitational field in interaction with dynamic electromagnetic and spinorial material field sources. The field equations are covariant under four-dimensional diffeomorphisms, local Lorentz transformations of the gravitational tetrad field, and U(1) gauge transformations.

He invented a constrained Hamiltonian procedure whose point of departure was the theorems of Noether and Klein (although the precise connection was never explicitly stated). It follows from these theorems that some linear combinations of conjugate momenta vanish identically.

Singular Legendre matrix

As a consequence of the identity (V) the Legendre matrix is singular, with null vectors ${c_{A\mu}}^0$.

$$0 \equiv \frac{\partial p^A}{\partial \dot{y}_B} {c_{A\mu}}^0 \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{y}_A \partial \dot{y}_B} {c_{A\mu}}^0$$

Suppose that $\mathcal{L}$ is quadratic in the field velocities. Then the full set of velocities cannot be solved uniquely in terms of the momenta

$$p^A = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \dot{y}_A \partial \dot{y}_B} \dot{y}_B + \ldots$$

Rosenfeld's Hamiltonian construction

First find a particular subset of solutions by working with a non-singular submatrix. He then proved that the momentum definitions are fulfilled when the remaining velocities vanish. Then he showed that the general solution was

$$\dot{y}_A(y,p) = {}_0\dot{y}_A(y,p) + \lambda^\mu(x) c_{A\mu}{}^0(y)$$

The λ^μ are arbitrary spacetime functions.

Rosenfeld's Hamiltonian construction

When inserted into $\mathcal{H} = p^A \dot{y}_A - \mathcal{L}$ one finds

$$\mathcal{H} = \mathcal{H}_0(y,p) + \lambda^A \phi_\mu(y,p)$$

Rosenfeld proved that the resulting Hamiltonian equations are equivalent to the original Euler-Lagrange equations.

Rosenfeld's symmetry generator

Simple! Use the Noether charge C^0!

$$C = \int d^3x \left(p^A \delta y_A - t^0{}_\nu \delta x^\nu \right) \qquad \text{(IIIB)}$$

There appeared no direct reference to either Noether or Klein - but Rosenfeld did identify $t^\mu{}_\nu$ as the "Impuls-Energie-Pseudo-tensor" and $t^0{}_\nu$ as the Impuls-Energie-Pseudodichte"

Rosenfeld proved that this canonical generator produced the correct infinitesimal variations of both the y and the p.

Secondary constraints

After proving that $\frac{dC}{dt} = 0$ - with no explicit mention of (IIIA) - Rosenfeld expanded C in factors of time derivatives of the arbitrary δx^μ, and found a recurrence relation for the vanishing coefficients. This determination of secondary and higher constraints has until now been attributed to Bergmann and Anderson 1951.

$$0 = C = \int d^3x \left(\phi_\mu \delta x^\mu_{,00} + \mathcal{N}^1_\mu \delta x^\mu_{,0} + \mathcal{N}^0_\mu \delta x^\mu \right)$$

4. PETER BERGMANN AND COLLABORATORS

Bergmann and particle equations of motion

Peter Bergmann was motivated through his relationship with Einstein and collaborators Infeld and Hoffmann to pursue a Hamiltonian formulation of the EIH derivation of particle equations of motion. He recognized that the contracted Bianchi identities were to play a crucial role. Although he did not in 1949 cite Noether explicitly in his first paper devoted to this topic, it is clear that he is referring to her second theorem in his statement of the theorem - in which no details are provided!

Bergmann and particle equations of motion

Note this later Bergmann quote from 1972:

"General relativity is characterized by the principle of general covariance, according to which the laws of nature are invariant with respect to arbitrary curvilinear coordinate transformations that satisfy minimal conditions of continuity and differentiability. A dsicussion of the consequences in terms of Noether's theorems (whether explicitly quoted as such or not) would have to include all of the work on ponderamotive laws, interalia . . . "
[Kimberling, 1972]

Bergmann and parameterized theory

Because of the desire to incorporate ponderamotive equations, Bergmann and collaborators initially pursued a parameterized approach in which spacetime coordinates themselves became dynamical variables. This complicating approach was dropped after Penfield 1951 [Penfield, 1951] realized that one could suspend with the parameters.

Bergmann and Brunings 1950 did consider how general coordinate transformations could be implemented in the parameterized context. They were the first to mention in print that not all variations in y, $\dot{y}$ space could be mapped onto phase space.

Bergmann et. al. construction of gravitational Hamiltonian

Bergmann, Penfield, Schiller, and Zatzkis 1950 [Bergmann *et al.* , 1950] were not aware of Rosenfeld's work. They proposed a method for solving the equations using "quasi-inverse" matrices. They did not cite a source. The procedure they used later came to be known as the Moore-Penrose construction for finding generalized inverses.

Anderson and Bergmann

The Bergmann group became aware of Rosenfeld's work in 1951. Anderson and Bergmann 1951 contains the introductory remark "This examination had begun in earlier papers, with particular emphasis on the type of covariance met with in the general theory of relativity, and is in some respects similar to the results obtained by L. Rosenfeld."

This paper contains an expression for a canonical generator of general coordinate transformations, expanded as with Rosenfeld in time derivatives of the arbitrary δx^μ.

$$\mathcal{C} = {}^2\mathcal{A}_\mu \delta x^\mu_{,00} + {}^1\mathcal{A}_\mu \delta x^\mu_{,0} + {}^0\mathcal{A}_\mu \delta x^\mu$$

Anderson and Bergmann

In requiring that this generator produce the correct variation in the Hamiltonian, they determined that the coefficients must vanish, and they found Poisson brackets that the constraints needed to satisfy. Rather than begin with Rosenfeld's Noether charge, they showed that the term multiplying the highest time derivative of δx^μ must be the primary constraints.

Ultimately Anderson and Bergmann obtained the same Noether charge as Rosenfeld - although they did not make this observation.

Bergmann and Schiller

With this information it is possible to show that $\bar{C}$ of (6.8) is actually the generator of the δy_A and the $\delta \pi^A$ transformations. The calculation is straight forward and closely follows a similar calculation in Rosenfeld,[13] so that we shall not carry it out. The only dissimilarity in Rosenfeld's and our work arises in the transformation law of the π^A, since $\delta \pi^A$ depends on the infinitesimal transformation properties of the Lagrangian density. Rosenfeld assumes that his action integral is an invariant, while ours differs from a scalar by a surface integral. This

From Schiiler's Ph. D. thesis, 1952

MEANWHILE - BACK IN CANADA . . .

5. PAUL DIRAC, PIRANI AND SCHILD

Dirac's motivation

Paul Dirac delivered his lectures on generalized Hamiltonian dynamics in Vancouver Canada in 1949. Pirani and Schild were in attendance. His motivation was to explore a means of rendering quantum electrodynamics manifestly Lorentz covariant by working with arbitrarily curved spacelike surfaces in flat spacetime. He was not aware of the usefulness of the method for general relativity until it was pointed out to him by Pirani and Schild.

In Dirac's typical idiosyncratic manner, he showed how one could construct the Hamiltonian for generally covariant systems by independently varying the y's, $\dot{y}$'s, and p's in the canonical Hamiltonian expression. [Dirac, 1950, Dirac, 1951]

Dirac's Hamiltonian

The result was a general expression for the Hamiltonian that was equivalent to that of Rosenfeld and of Bergmann et. al., but which did not require the same sophisticated mathematical justification.

Probably the most significant contribution for later use in general relativity was Dirac's realization that a technical advantage was to be gained by giving a central role to the projection of canonical momenta in the direction perpendicular to the spatial hypersurfaces. He likely borrowed this insight from his thesis student (co-directed by Max Born) - Paul Weiss. [Weiss, 1938]

Pirani and Schild almost immediately set to work to construct the Hamiltonian for Bergmann's parameterized system, beating the Bergmann group to publication in the same year, 1950 [Pirani & Schild, 1950]. They employed the Weiss insight.

Dirac and general covariance

Dirac never investigated the possible realization of general covariance in phase space - even following his simplification of the gravitational Hamiltonian in 1958 [Dirac, 1958]. He exhibited no equivalent of the Noether charge - and indeed his attitude toward the nature of gauge generators is debated even today.

6. ADM AND KUCHAŘ

ADM

I cannot do justice to the enormous contribution of Arnowitt, Deser, and Misner in this short talk. The comprehensive history is coming! Suffice it to say that the stress-energy pseudo tensor did play a role in the papers they wrote in the period from 1959 through 1962 [Arnowitt *et al.* , 1962]. But the significant observation for this talk is that they shared with Dirac a similar dismissal of a role for general covariance in the constrained Hamiltonian approach to classical general relativity.

Deser's philosophy

SALISBURY: Could you say a little bit about what your motivation was in pursuing this self-interaction?
DESER: Oh yes. That was, again, an anti-geometry reaction. Dick and I (Charlie, of course, had his foot in both camps) were strongly anti-geometrical, as we felt that Riemann was clearly a corrupting influence on Einstein . . . Remember, this was not meant to be a breakthrough beyond GR. It was really a point of honor that I felt I owed myself to understand the theory. If I'm so bitterly anti-Riemann, put up and show me how else you could understand it.
(Interview DS and Dean Rickles with Stanley Deser, March 12, 2011)

Kuchař

Nor can I do justice to Karel Kuchař's important work. In following the lead of ADM he rejected the notion that the full diffeomophism symmetry group could be implemented as a canonical transformation group. Rather, he introduced a notion of "multi-fingered time" [Kuchař, 1972]

These views of Dirac and Kuchař have in the last forty years become conventional wisdom in the relativity community.

BACK TO SYRACUSE . . .

7. BERGMANN AND KOMAR

D-invariants

In his 1962 Handbuch der Physik [Bergmann, 1962] article Bergmann emphasizes the special role played by the perpendicular components of the momenta conjugate to the gravitational 3-metric. He points out that the Lie algebra formed by the descriptors ${}_1\delta x^\mu$ and ${}_2\delta x^\mu$ of two infinitesimal diffeomorphisms necessarily contains time derivatives,

$${}_1\delta x^\mu_{,\nu}\,{}_2\delta x^\nu - {}_2\delta x^\mu_{,\nu}\,{}_1\delta x^\nu = {}_1\delta x^\mu_{,0}\,{}_2\delta x^0 - {}_2\delta x^\mu_{,0}\,{}_1\delta x^0 + \ldots$$

Indeed the algebra yields time derivatives of infinite order! However, observes Bergmann, the variations of Dirac's momentum variables do not involve time derivatives. In fact, if one considers diffeomorphisms that leave a spatial hypersurface fixed and change coordinates only off the hypersurface, then Dirac's variables are invariant. Hence Bergmann coined the term "D-invariant" for such variables.

D-invariants

The implications for Bergmann were profound. Since the lapse and shift variables of general relativity were decidedly not D-invariant, they were discarded as phase space variables. Thus the Noether charge C was abandoned as a generator of symmetry transformations.

D-invariants

Bergmann followed Dirac's lead in considering infinitesimal coordinate transformations that were either tangent to constant time hypersurfaces or perpendicular to them, i.e., $\delta x^\mu = n^\mu \xi^0 + \delta^\mu_a \xi^a$. But he then eliminated the lapse and shift as canonical variables, retaining only $\bar{\delta} g_{ab}$ in the Noether generator (IIIB)

The Bergmann-Komar group

On the other hand, Bergmann and Komar were anxious to understand the group theoretical significance of the Dirac algebra that replaced the Lie algebra in the Hamiltonian formulation of GR. They concluded in 1972 [Bergmann & Komar, 1972] that the diffeomorphism group had been replaced by a phase space transformation group that depended on the 3-metric. Indeed, they argued that the group elements depended spatially non-locally on the metric because of the appearance of spatial derivatives up to infinite order in the Poisson bracket algebra.

Castellani generator

Castellani 1982 "Symmetries in Constrained Hamiltonian Systems" [Castellani, 1982]

8. NOETHER CHARGE REGAINED

The Noether charge regained

OK - I will now "cut to the chase". The condition that we must satisfy in order to construct finite elements of the Bergmann-Komar group is that only Legendre-projectable variations of the configuration-velocity variables may be permitted in the Noether charge. (Josep Pons, DS, Larry Shepley 1997) [Pons *et al.* , 1997]

For example, in Einstein-Yang-Mills theory the permissible hypersurface altering diffeomorphisms are $\delta x^\mu = -n^\mu \xi^0$, where $n^\mu = \left(N^{-1}, -N^{-1} N^a\right)$ is the normal to the hypersurface. In addition one must perform a gauge transformation with descriptor $A^i_\mu n^\mu \xi^0$. Then the Noether charge is

$$C = \int d^3x\, C^0 = \int d^3x \left(p^{ab} \bar{\delta} g_{ab} + p_\alpha \bar{\delta} N^\alpha + p^\alpha_i \bar{\delta} A^i_\alpha + \mathcal{L}_{EYM} N^{-1} \xi^0 \right) \quad \text{(IIIB)}$$

The Noether charge regained

It was shown in (Pons, DS, Shepley 2000) [Pons *et al.* , 2000] that this object generates the correct phase space transformations. We did not realize at the time that this was the Rosenfeld - Bergmann Noether charge!

8. CONCLUSIONS

Conclusions

- The vanishing Noether charge associated with the general covariance of Einstein's equations has served as the canonical generator of infinitesimal associated phase space variations since Rosenfeld's pioneering work in 1930
- Bergmann and collaborators in the early 1950's rediscovered and extended the use of this charge - though then partially abandoned it after Dirac's breakthrough
- After Dirac's breakthrough in which general covariance did not constitute a foundational principle, ADM and Kuchar promoted a formalism in which the full four-dimensional diffeomorphism covariance was abandoned.

Conclusions

- Low water mark?

Conclusions

- Low water mark? No

Conclusions

- Low water mark? No
- étiage?

Conclusions

- Low water mark? No
- étiage? Peut-être

Conclusions

- Low water mark? No
- étiage? Peut-être
- Renaissance?

Conclusions

- Low water mark? No
- étiage? Peut-être
- Renaissance?
- No

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