Research Topics

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# Johann Lambert's Conversion to a Geometry of Space

## Steps towards a new concept of geometry in the 18th century

By Vincenzo De Risi

In the eighteenth century two very different conceptions of geometry fought for prominence in Germany and Europe. The first of them relied on the classical definition of geometry as a science of continuous magnitudes and figures, a science of triangles and squares, circles and conic sections, parallel lines and obtuse angles; this conception of geometry was as old as Aristotle and Euclid, and was maintained by mathematicians and philosophers in the whole of Antiquity, in the Islamic Middle Ages, in the Italian *Quattrocento* and in the late European Renaissance from Clavius to Newton.

But then, however, a second concept of geometry had begun to appear in some daring essays on perspective and parallel theory: the idea that geometry could be the science of *space itself*, and that space, might be endowed with a geometrical structure more fundamental than any triangle or figure that might be described in it. In this new theory, space did not anymore constitute an amorphous background field, a sort of conceptual (or imaginary) arena in which geometry proper enacts its straight-edge-and-compass constructions, but takes the form of a geometrical object itself with its own properties.

This last, modern and far-reaching conception of geometry was mainly envisaged by Gottfried Wilhelm Leibniz (1646-1716), who developed it in hundreds of essays, both philosophical and mathematical, as part of a geometrical *analysis situs*. The mathematical results of this project were not published in his lifetime, nor even during the course of the 18th century. Leibniz's philosophical arguments and epistemological assumptions, however, concerning the necessity of a proper geometry of space did gain some adherents. In Germany, the new "Leibnizean" school of Christian Wolff (1679-1754) and his followers put in widespread circulation the idea that geometry has to be a science of space. The main problem in this connection was that Wolff was by far too poor a mathematician to rediscover what Leibniz had fashioned in forty years of geometrical investigations but which then lay buried in the library of Hannover. Wolff *declared* that geometry was the science of space, and proposed that a careful analysis of the very concept of space would produce a rich field in which one could ground the whole of geometry; but then he proceeded to offer the usual Euclidean definitions and axioms (of lines and circles, not space), and to construct with straight-edge and compass, one by one, the usual theorems and problems of the *Elements*. The geometry of space thus remained an idea devoid of any real content. Later in the century, even the well-known Leibnizean and anti-Kantian polemist Johann Augustus Eberhard (1739-1809) had to admit that Leibniz's definition of geometry as a science of *situs* and space was virtually useless in mathematics.

In the beginning at least, then, the "conservative wing" of German mathematicians and philosophers, who continued to regard geometry as a science of magnitudes and figures, and who viewed space as the shapeless and ungeometrical container of proper mathematical objects, won an easy victory.

Among them, the story of Johann Heinrich Lambert (1728-1777) is especially revealing. His beginnings as a mathematician and an epistemologist clearly show that he began his philosophical investigations about mathematics in the most classical fashion, and that he rejected the Wolffian conception of space to embrace the Euclidean one. His first major philosophical work, the *Neues Organon* from 1764, defended the conception that the idea of space is *simple* (uncomposed) and thus cannot possibly be analyzed; so that the very conception of geometry as an analysis of space was decidedly ruled out. The following year (on April 21st) Lambert wrote to Georg Jonathan Holland that geometry was correctly carried out by Euclid not as an *Analysis* of the concept of space but as an *Anatomy* of it. The correct procedure of the mathematician, he asserted, is to take the simple idea of extension and cut it into pieces (geometrical figures), which would in their turn display all the geometrical properties that space itself lacked. Later in 1765 (on November 13th) he wrote to Kant that Wolff had completely misconceived Euclid. On February 3rd, 1766, he reasserted in another letter to Kant that the concept of space is simple, and that it nowhere appears in the whole of Euclid's *Elements* – and rightly so, since it is in fact useless.

But then, abruptly, Lambert ended his bold declarations. He stumbled, in fact, on a short essay (from 1763) by the mathematician Georg Simon Klügel (1739-1812), in which a number of attempts to prove the Parallel Postulate were described in detail. Lambert became fascinated by the topic and began to study the matter more deeply, eventually giving rise to his own at-

The mathematician Luca Pacioli (1495) illustrating the geometry of proportions. Portrait by an unknown painter. Source: Wikimedia Commons.

tempt at a demonstration of the famous axiom. His *Theorie der Parallellinien* was (presumably) penned the following September (1766) but ended in unmistakable failure. Lambert never published it, and had to resign himself to the weaker position in which the Parallel Postulate stands as an indemonstrable principle of the whole of geometrical science. Nevertheless, such failure had something to teach the epistemologist.

One of the major breakthroughs in the theory of parallelism was John Wallis's (1613-1703) essay *De Postulato Quinto* from 1663 (first published in 1693), which Lambert probably came to in connection with his own attempt to prove the Euclidean axiom. Wallis had demonstrated that the assumption of the Parallel Postulate was equivalent to the possibility of transforming any figure by similarity; given a triangle, for example, it is possible to construct a similar triangle (i.e. a triangle with equal angles and smaller or larger proportional sides) if and only if the Parallel Postulate holds. With his characterization theorem Wallis claimed to have *proved* the Euclidean axiom itself, because he thought he had *metaphysical* reasons to hold that any figure can be transformed in *quantity* (enlarged or shrunk in size) without be changed also in *quality* or shape; and thus he asserted that the principle of similarity was unquestionably true, and thus (thanks his theorem) the Parallel Postulate itself.

The learned reactions to Wallis's argument varied; but, in any case, it clearly demonstrated that the Parallel Postulate, which in Euclid's wording was an axiom about straight lines and angles, is in fact concerned with something much more abstract and unfamiliar. It concerns the possibility of certain transformations in space, which affect all possible figures and magnitudes; it ultimately amounts to a true axiom about the structure of space itself. This last understanding was probably the greatest advance that Wallis's work on parallelism produced in eighteen-century geometry.

It is easily understandable that Lambert had to regard this result as one of great epistemological relevance. His attempt to demonstrate the Parallel Postulate with purely mathematical arguments was probably also an attempt to defuse its philosophical significance. When he failed, and realized the implications of his failure, he became well aware that it was simply impossible for him to regard the Euclidean axiom as a principle about straight lines: it had revealed itself to him as something about the structure of space and transformations, and he could not anymore regard "extension" as a shapeless background field, or a "simple idea" devoid of geometrical content.

In a later work published in 1771, the *Analage zur Architektonik*, Lambert was forced to provide geometrical axioms about space itself, not just about figures or magnitudes. The second of these is exactly the Parallel Postulate that he failed to prove, in Wallis's form: *Der Raum hat keine bestimmte Einheit*… He had thus surrendered to Leibniz's idea of a geometry of space, not by simple philosophical arguments, but through a painful (and beautiful) attempt to prove a particular mathematical result.

The story of Lambert "conversion" from Euclid to Leibniz is very telling about the relationships between philosophy and mathematics in the eighteen century, and is an important event in the gradual transformation of classical geometry into a modern theory of spaces. Other thinkers, both mathematicians and philosophers, in the very same years but on completely different grounds, arrived to the same result; and the generation after Lambert would come to regard the definition of geometry as a science of space as so obvious there was no need even to explain it.

The aims of the research group on *Modern Geometry and the Concept of Space* will include pursuing an investigation of Lambert's role in the conceptual developments of geometry in the eighteenth century and exploring how other figures with other ambitions converged in the same years to consolidate the same momentous results.

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